Convolution in Neural Networks

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References: https://github.com/vdumoulin/conv_arithmetic
Justin Johnson, Bernhard Kainz, Jon Hare

Motivation

- So far, we have focused on MLPs
- How is a 2D image input into an MLP?
- How can we keep the spatial information?

n02097047 (196)

n01682714 (40)

n03134739 (522)

n04254777 (806)

n02859443 (449)

n02096177 (192)

n02107683 (239)

n01443537 (1)

n02264363 (318)

Components of a Convolutional Neural Network (CNN)

- **Convolution Layers**
- Activation Functions
- **Pooling Layers**
- **Normalization**
- Fully Connected Layers

The Convolution Operation

In the **time domain**, convolution is:

$$
\begin{aligned}
(f * g)(t) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau.
\end{aligned}
$$

Notice that the image or kernel is "flipped" in time, where f is the image and g is the kernel.

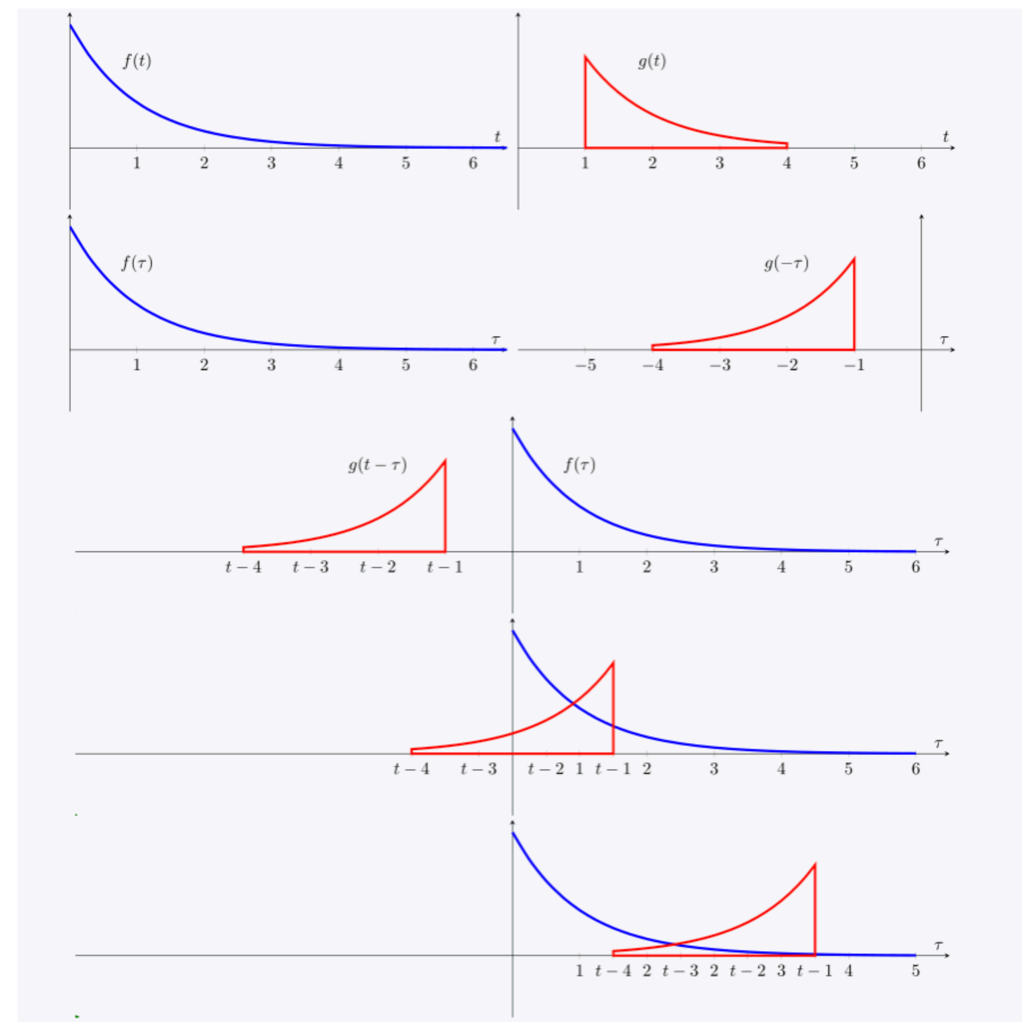


Image taken from: Wikipedia

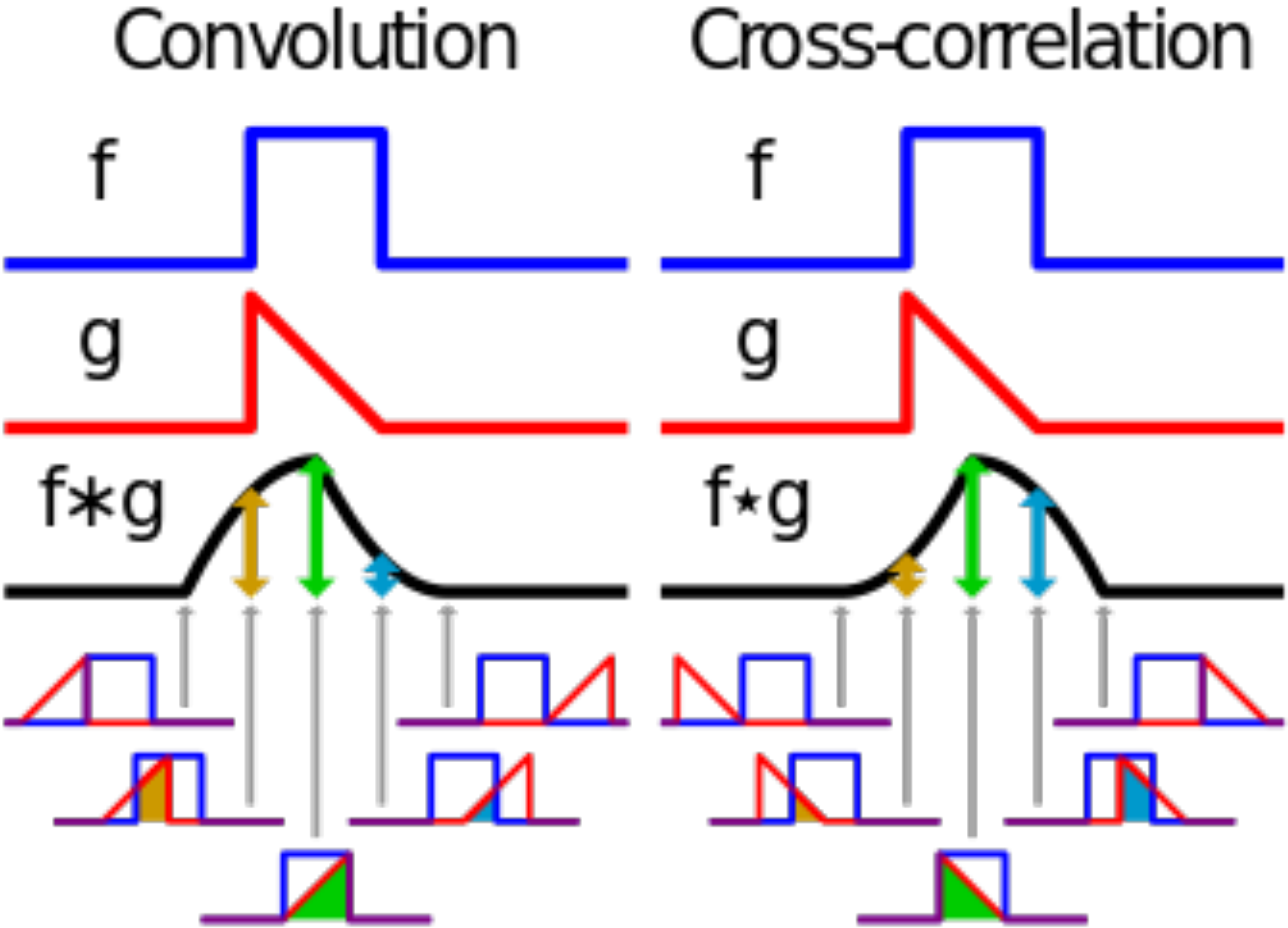


Image taken from: Wikipedia

Cross-Correlation in Practice

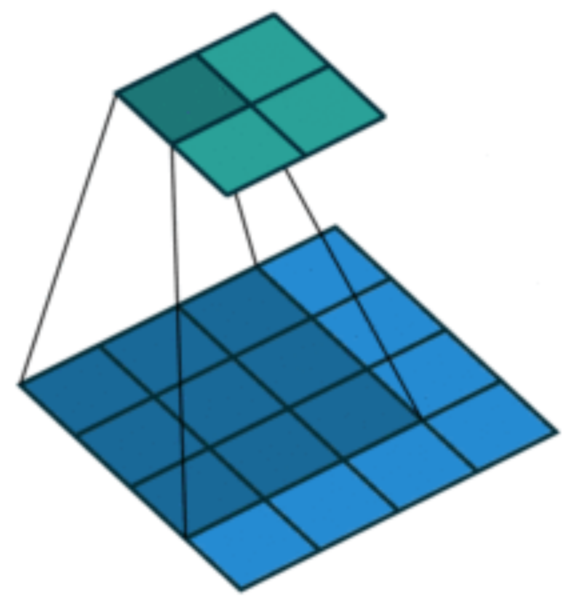
Convolution over a two-dimensional input image I and two-dimensional kernel K is defined as:

(1) $$S(i, j) = (I^*K)(i, j) = \sum_m \sum_n I(i-m, j-n)K(m, n)$$

However, nearly all machine learning and deep learning libraries use the **simplified cross-correlation function**

(2) $$S(i, j) = (I^*K)(i, j) = \sum_m \sum_n I(i+m, j+n)K(m, n)$$

Convolution Visualised



Visual link https://github.com/vdumoulin/conv_arithmetic/blob/master/gif/no_padding_no_strides.gif

"Convolution" in Neural Networks

- "Convolution" in the neural network literature almost always refers to an operation akin cross-correlation
  - An element-wise multiplication of learned weights across a receptive field, which is repeated at various positions across the input.
- Normally, we also add an additional *bias term*; a single bias term for each *kernel.*
- There are also other parameters of these "convolutions"…

Convolutional Layers

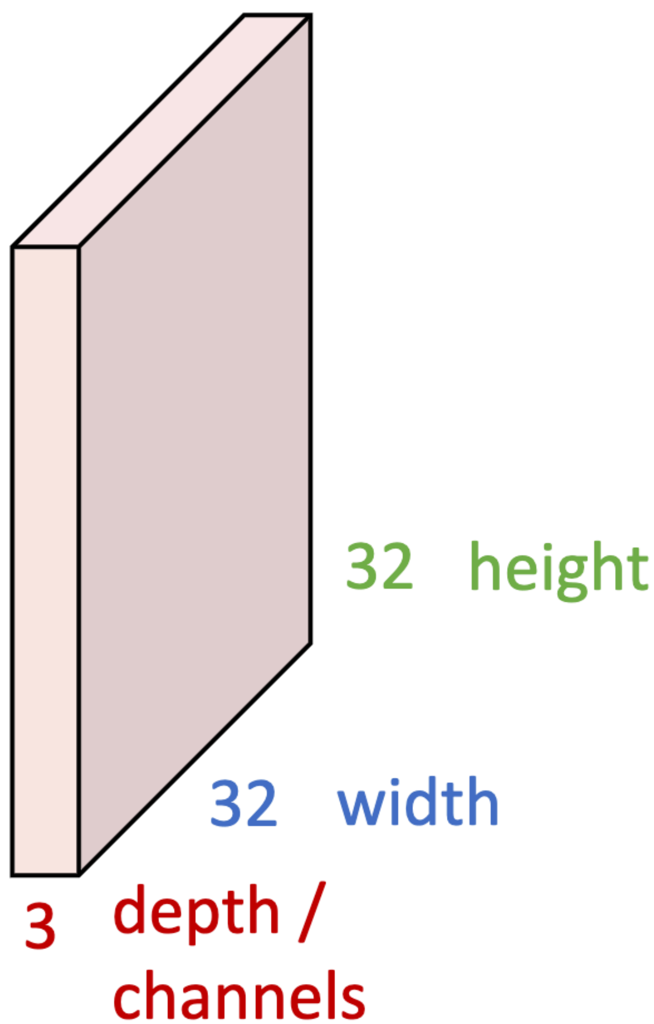
- In a convolutional layer, we have multiple kernels or filters which are learnt (plus the biases). This set of kernels can be called a bank of kernels.

- Each filter produces a single “Response Map” or “Feature Map” or "Activation Map". The activation maps are stacked together as “channels” of the resultant output tensor

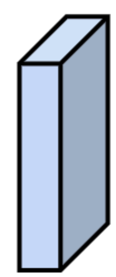
- Each activation map tells us how much does each position in the input respond to the corresponding convolutional filter

Convolution Layer

3x32x32 image

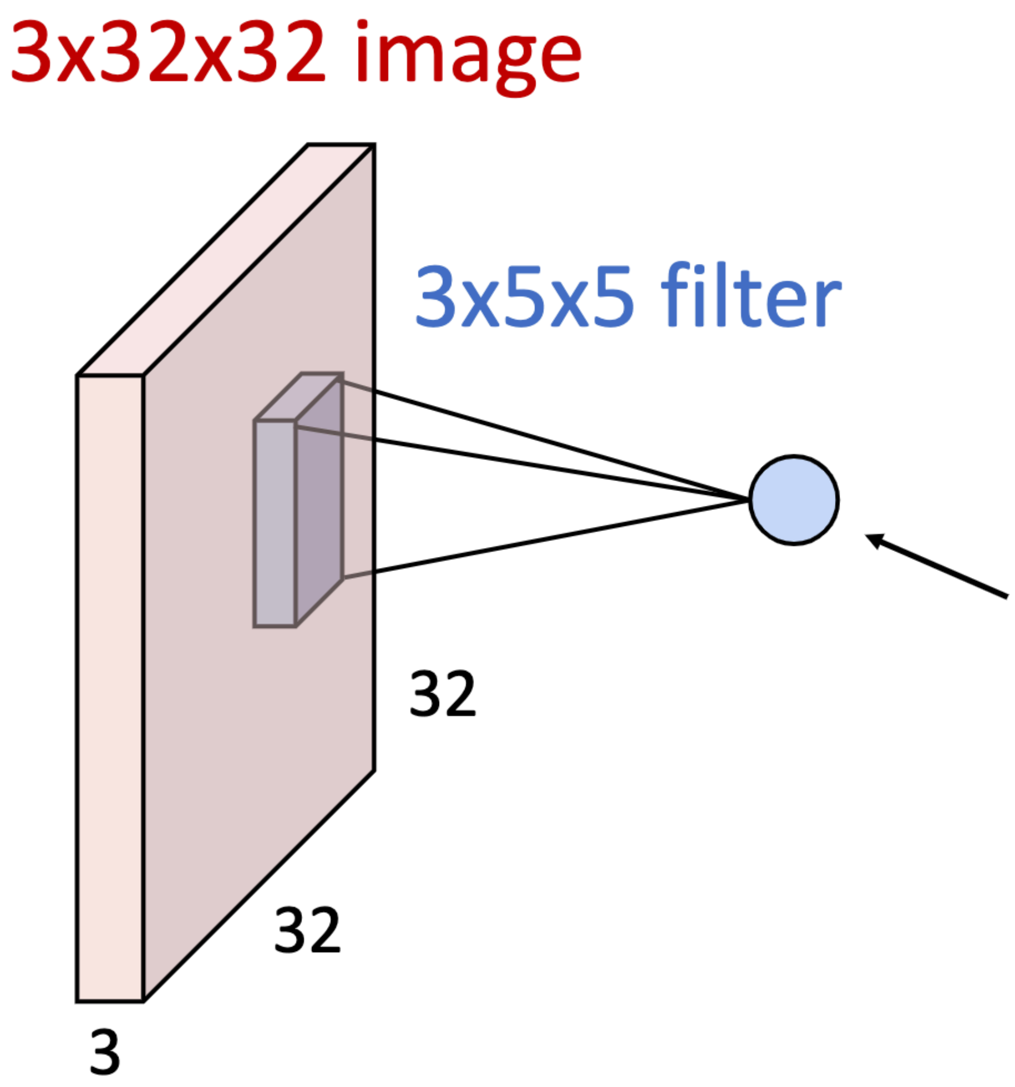


3x5x5 filter



Convolve the filter with the image i.e. “slide over the image spatially, computing dot products”

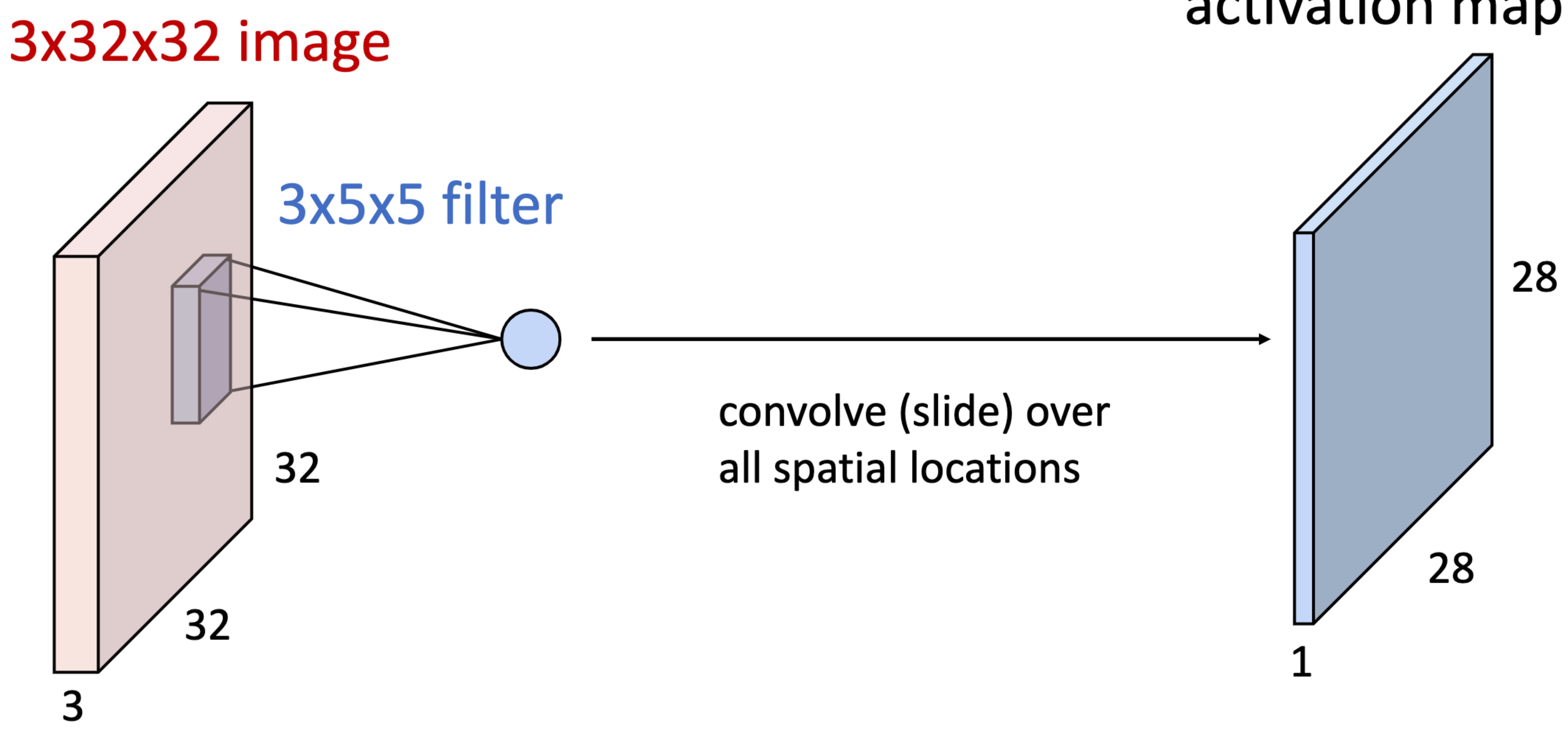
Convolution Layer



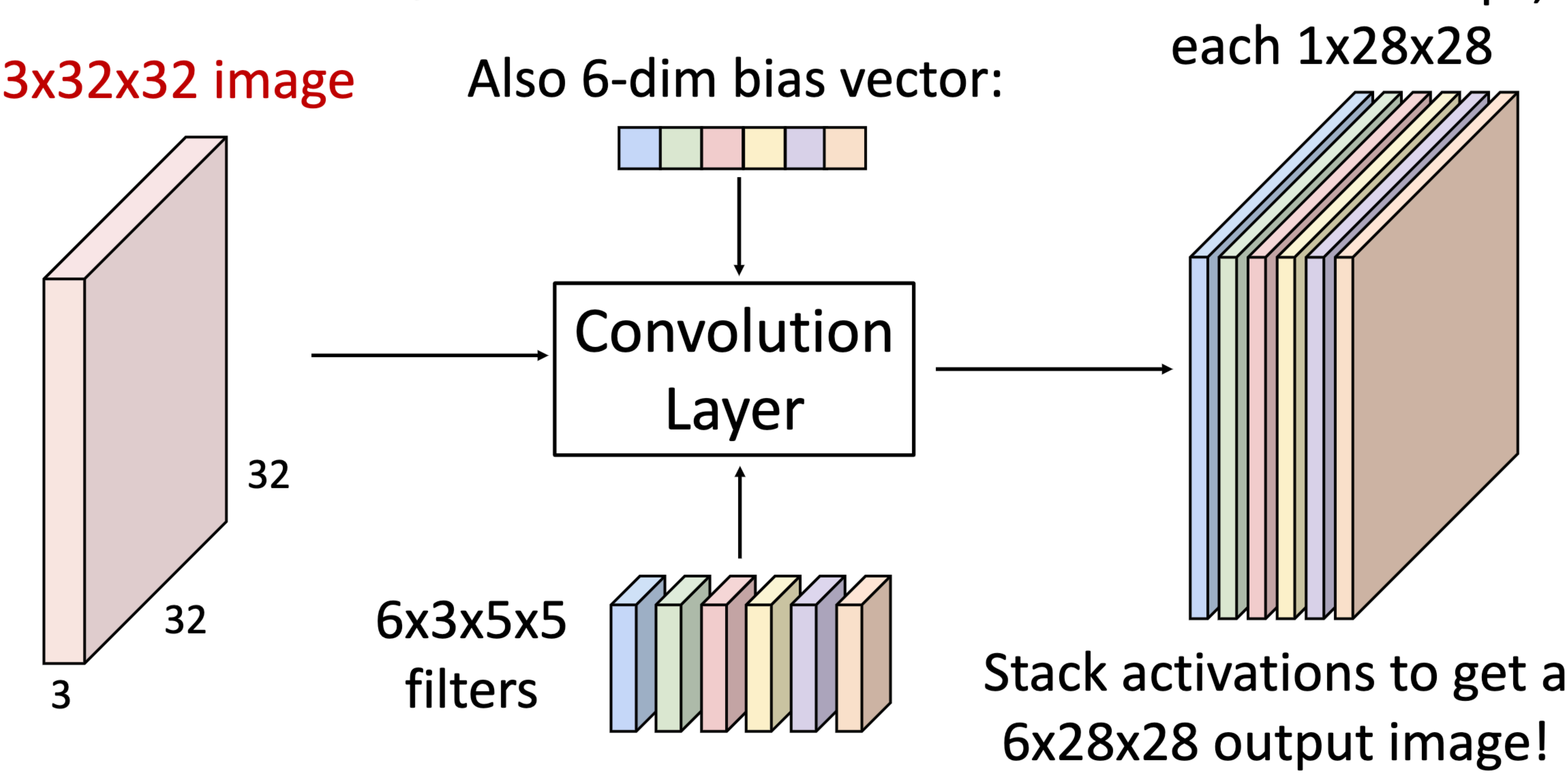
1 number:
the result of taking a dot product between the filter and a small 3x5x5 chunk of the image
(i.e. 3*5*5 = 75-dimensional dot product + bias)

$$w^T x + b$$

Convolution Layer



Convolution Layer



Convolution as a Matrix Multiplication

- The convolution operation can be expressed as a matrix multiplication if either the kernel or the signal is manipulated into a form known as a Toeplitz matrix:

$$
y = h * x = \begin{bmatrix}
h_1 & 0 & \dots & 0 & 0 \\
h_2 & h_1 & \dots & \vdots & \vdots \\
h_3 & h_2 & \dots & 0 & 0 \\
\vdots & h_3 & \dots & h_1 & 0 \\
h_{m-1} & \vdots & \dots & h_2 & h_1 \\
h_m & h_{m-1} & \vdots & \vdots & h_2 \\
0 & h_m & \dots & h_{m-2} & \vdots \\
0 & 0 & \dots & h_{m-1} & h_{m-2} \\
\vdots & \vdots & \vdots & h_m & h_{m-1} \\
0 & 0 & 0 & \dots & h_m
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n
\end{bmatrix}
$$

- For 2D convolution one would use a “doubly block circulant matrix”

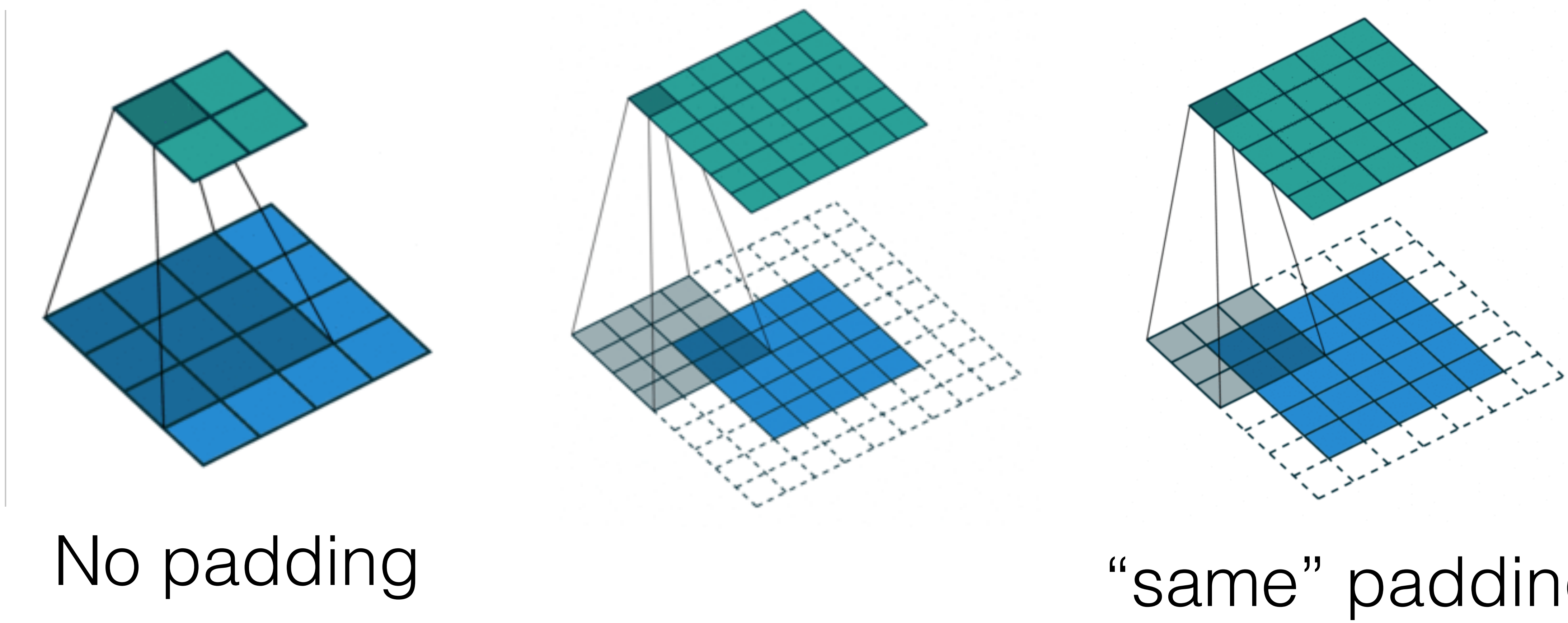
2D Convolutions with kernel size of 1

- 1x1 convolutions are a common place operation, but might seem non-sensical at first
  - They do not capture any local spatial information
  - They are used to change the number of feature maps without affecting the spatial resolution

Padding

- What happens to a convolution at the edges of its spatial extent?
- In spatial convolution if we do nothing, the output will be smaller…
- We often use zero-padding to retain the size
- With "same" padding, $P = (K-1) / 2$ to make the output the same size as the input
- Output: $W - K + 1 + 2P$ (K = kernel size, W = input size)

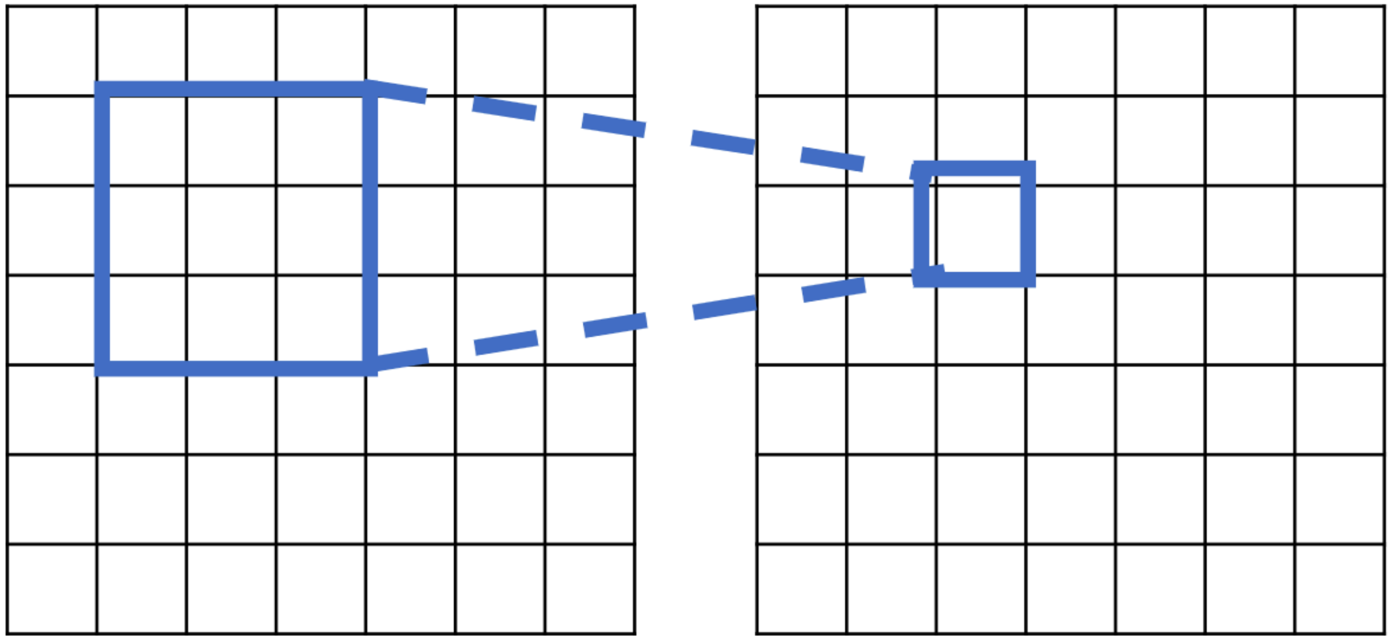
Arbitrary padding



https://github.com/vdumoulin/conv_arithmetic/blob/master/gif/arbitrary_padding_no_strides.gif

Receptive Fields

For convolution with kernel size K, each element in the output depends on a K x K **receptive field** in the input

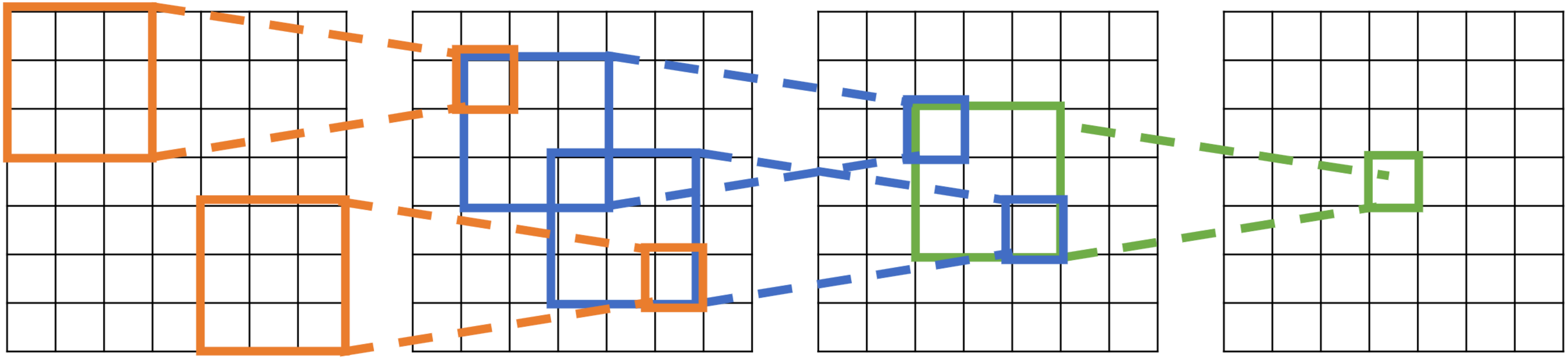


Input

Output

Receptive Fields

Each successive convolution adds K – 1 to the receptive field size
With L layers the receptive field size is 1 + L * (K – 1)

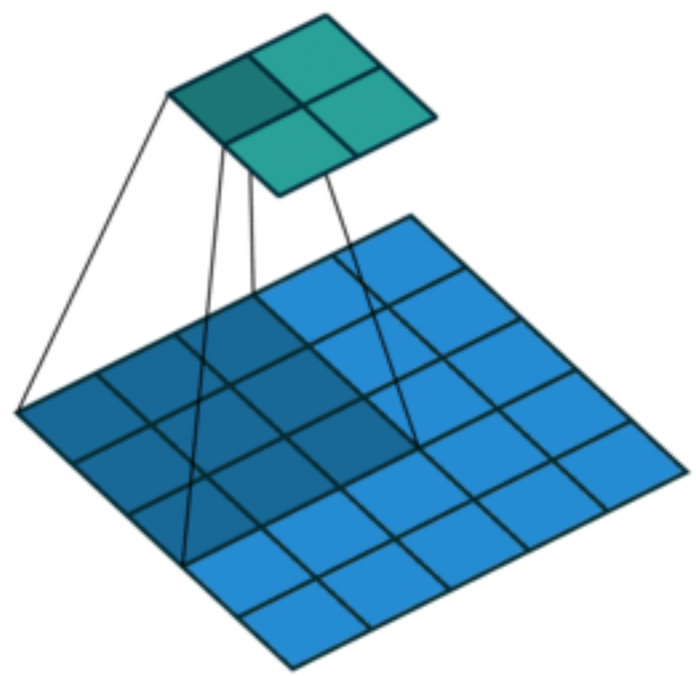


Input

Output

Striding

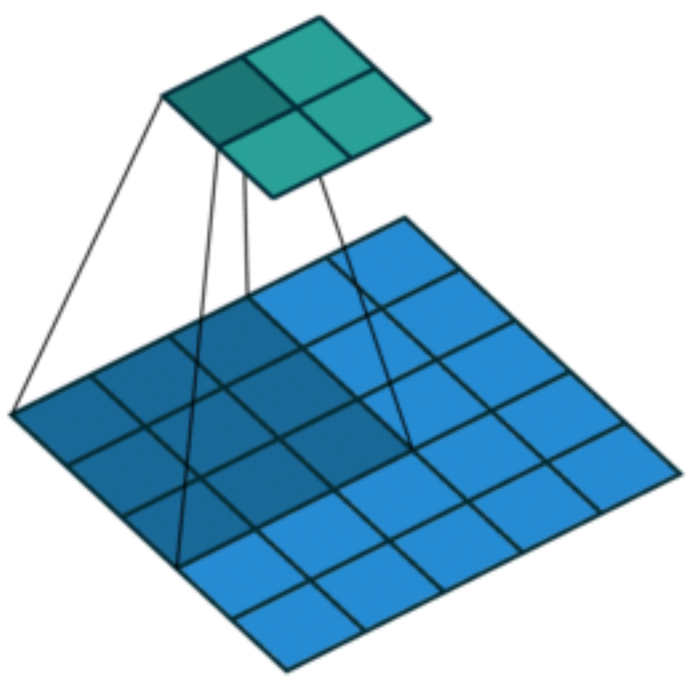
- Convolution is expensive… could we make it cheaper by skipping over positions?



Stride=(2,2)

Striding

- Convolution is expensive… could we make it cheaper by skipping over positions?



Stride=(2,2)

In general:
Input: W
Filter: K
Padding: P
Stride: S
Output: (W – K + 2P) / S + 1

https://github.com/vdumoulin/conv_arithmetic/blob/master/gif/no_padding_strides.gif

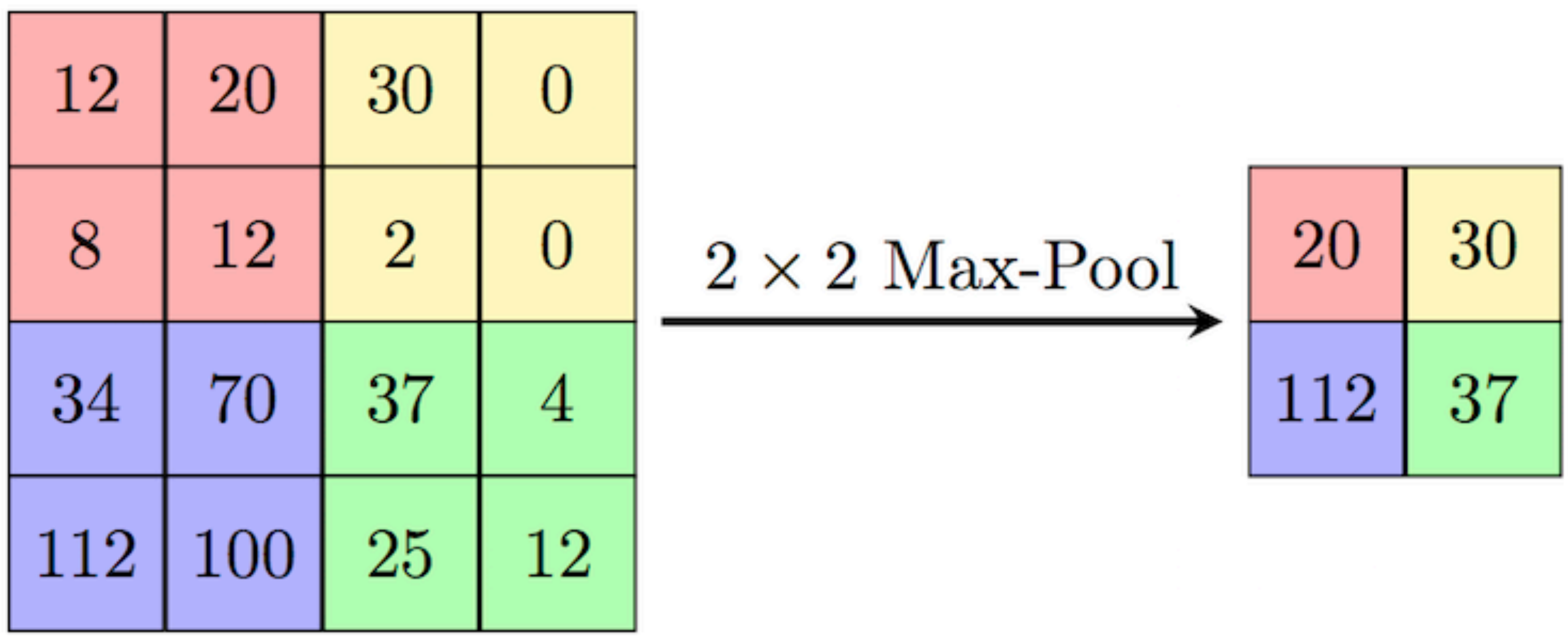
Fractional Striding/ Transpose Convolution

- What if we consider *fractional* strides between 0 and 1?
  - Intuitively, if bigger strides subsample, then fractional strides should upsample
  - This is equivalent to “expanding” the input by padding and performing convolution
    - And potentially also striding by adding zeros around all the values

Pooling

- Striding is a popular way to reduce spatial dimensionality in modern networks
- Before striding was devised, **pooling**, was the defacto way of reducing dimensionality
- Pooling reduces the number of parameters to learn and the amount of computation performed in the network
- The pooling layer summarises the features present in a region of the feature map

Max Pooling, 2x2, stride=2

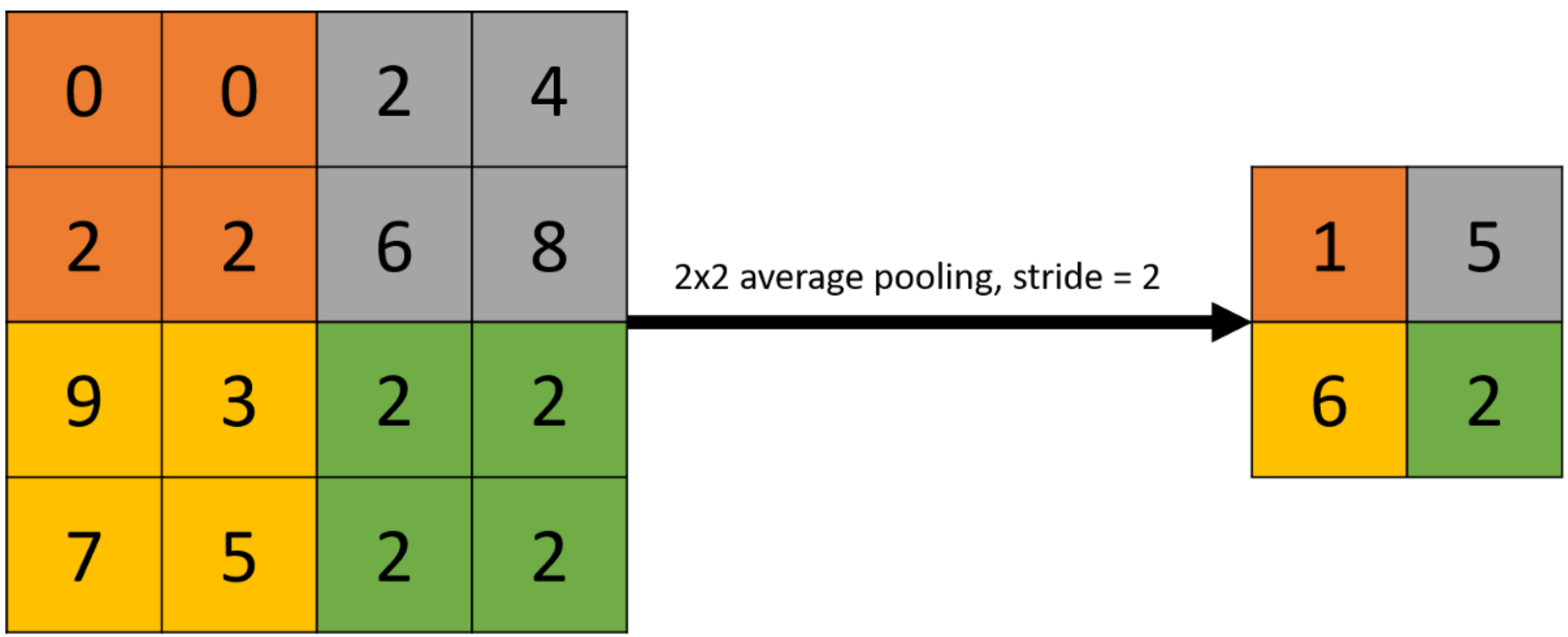


Note: The default stride for 2D max pooling in PyTorch is the kernel size

Max Pooling Gradients

- The gradient of the max pooling operation is 1 everywhere a max value was selected, and zero elsewhere
- This means that implementations not only need to record the max values in the forward-pass, but also keep track of the positions of those maximums for the backward pass

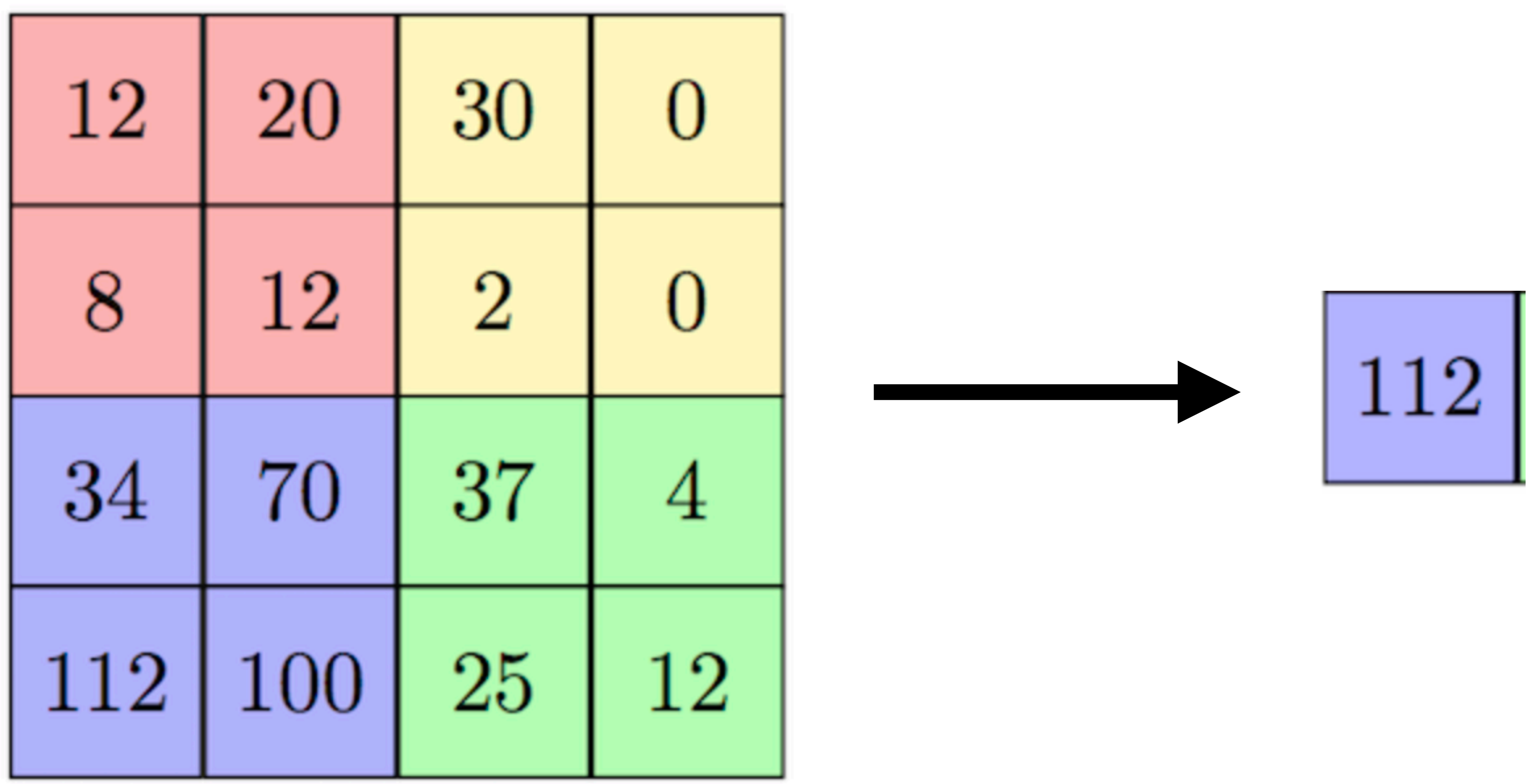
Average Pooling



Local Versus Global Pooling

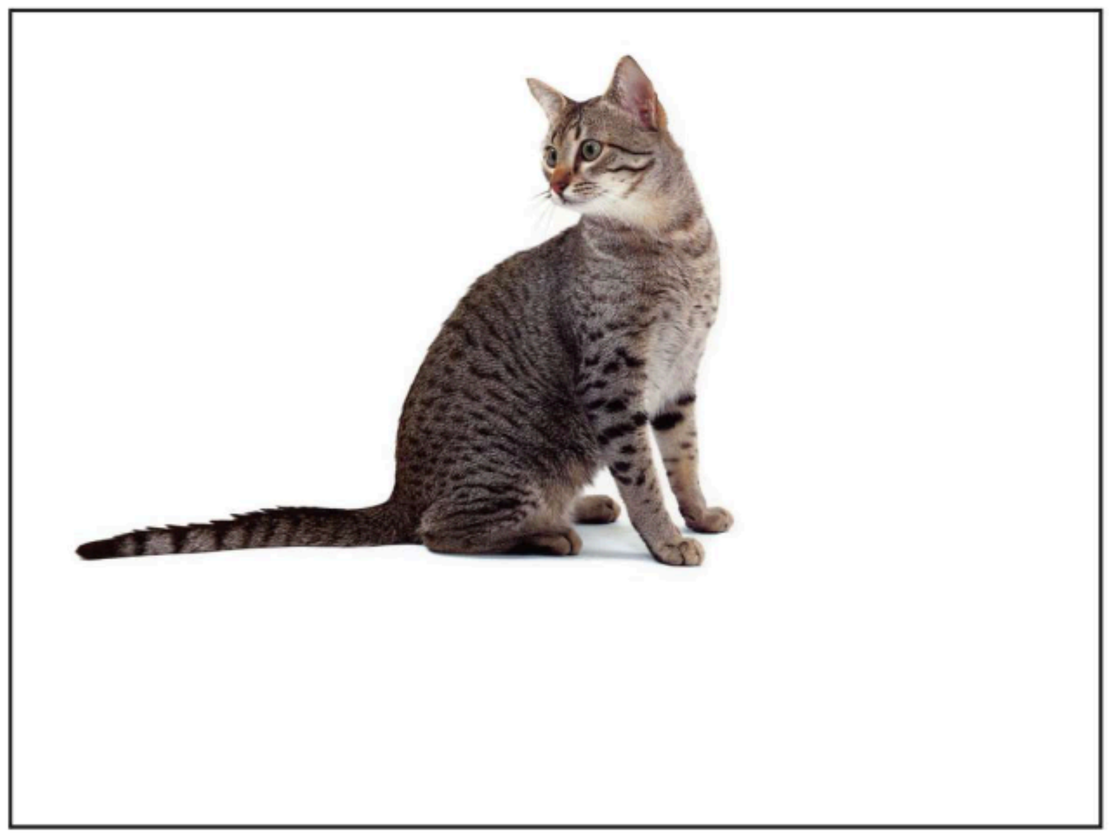
- The pooling operations on the previous slides are local
    - They result in a feature map reducing in spatial size
- Global pooling reduces a feature map to a scalar
    - So a tensor of many feature maps would be reduced to a single feature vector
    - Often used near the end of networks to flatten feature maps into feature vectors that can be fed into an MLP

Global Max-Pooling



Invariance and equivariance

- **Shift invariance**



'cat'

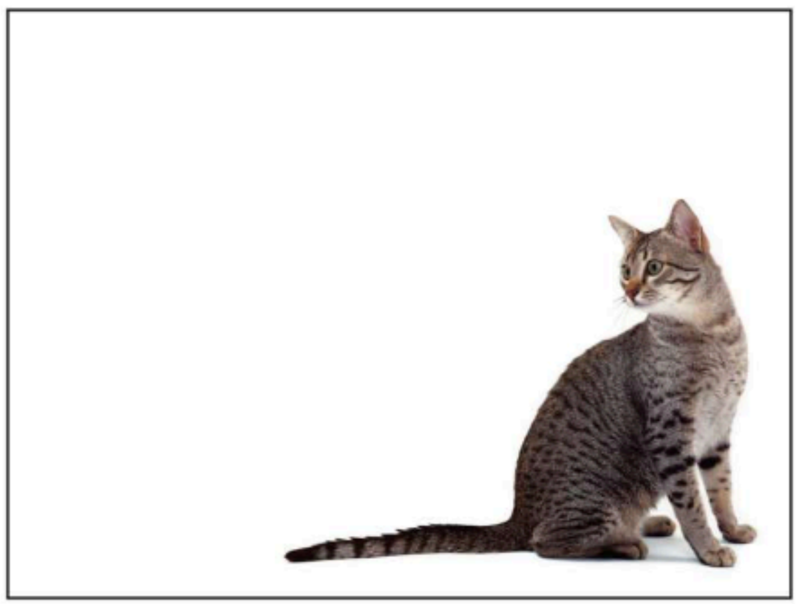
Invariance and equivariance

- **Shift invariance**

'cat'

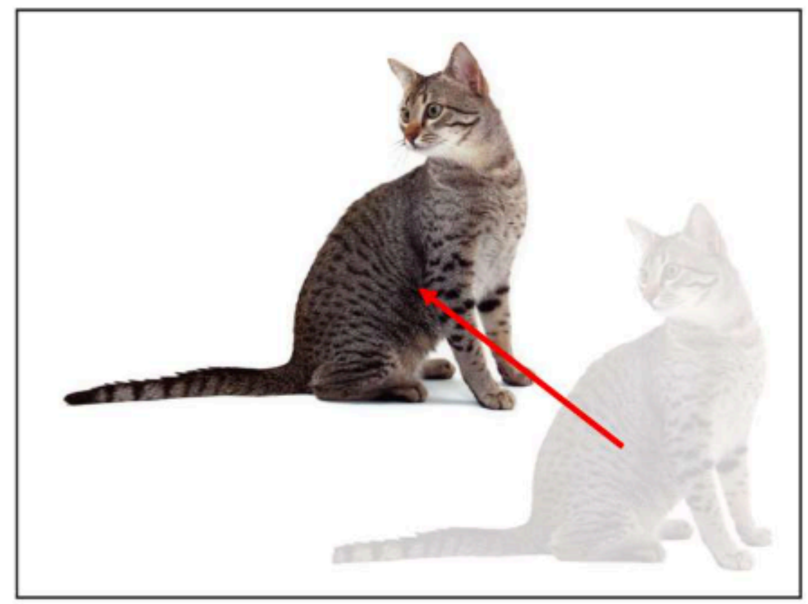
Shift invariance

Input $\mathbf{x}$



Output $f(\mathbf{x}) = 1$

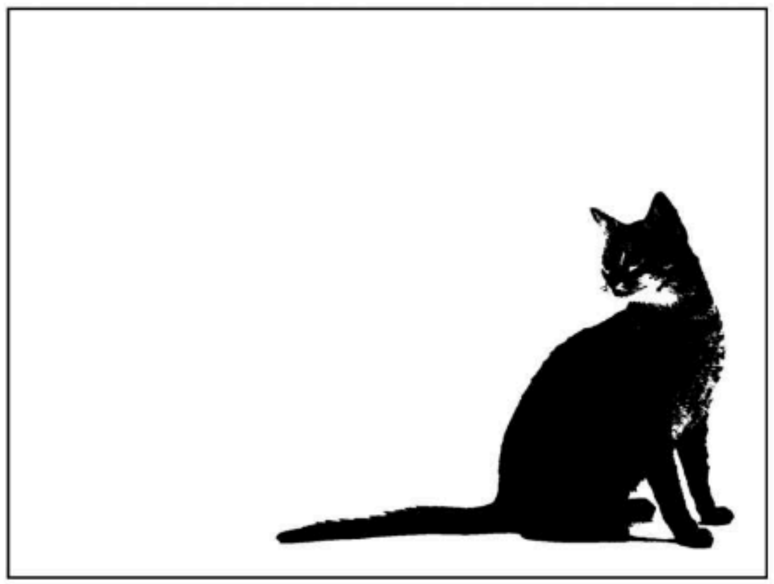
Shifted input $S_v\mathbf{x}$



Output $f(S_v\mathbf{x}) = 1$

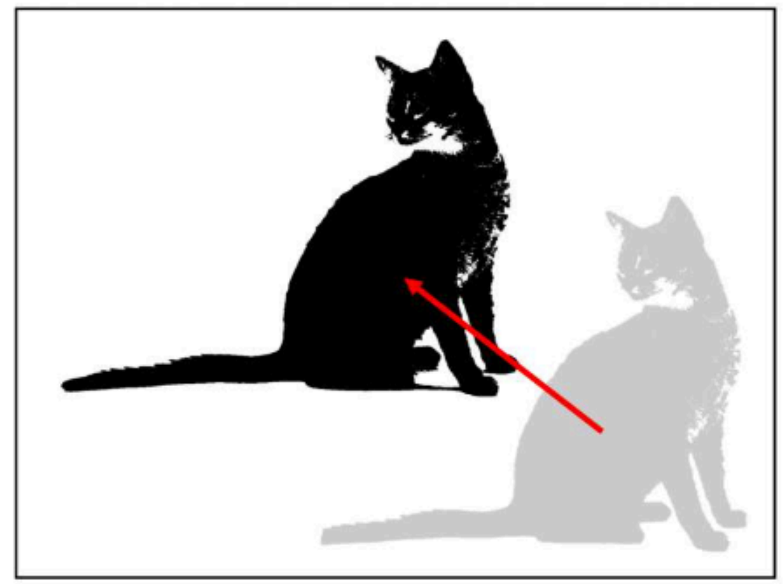
Shift equivariance

Input $\mathbf{x}$



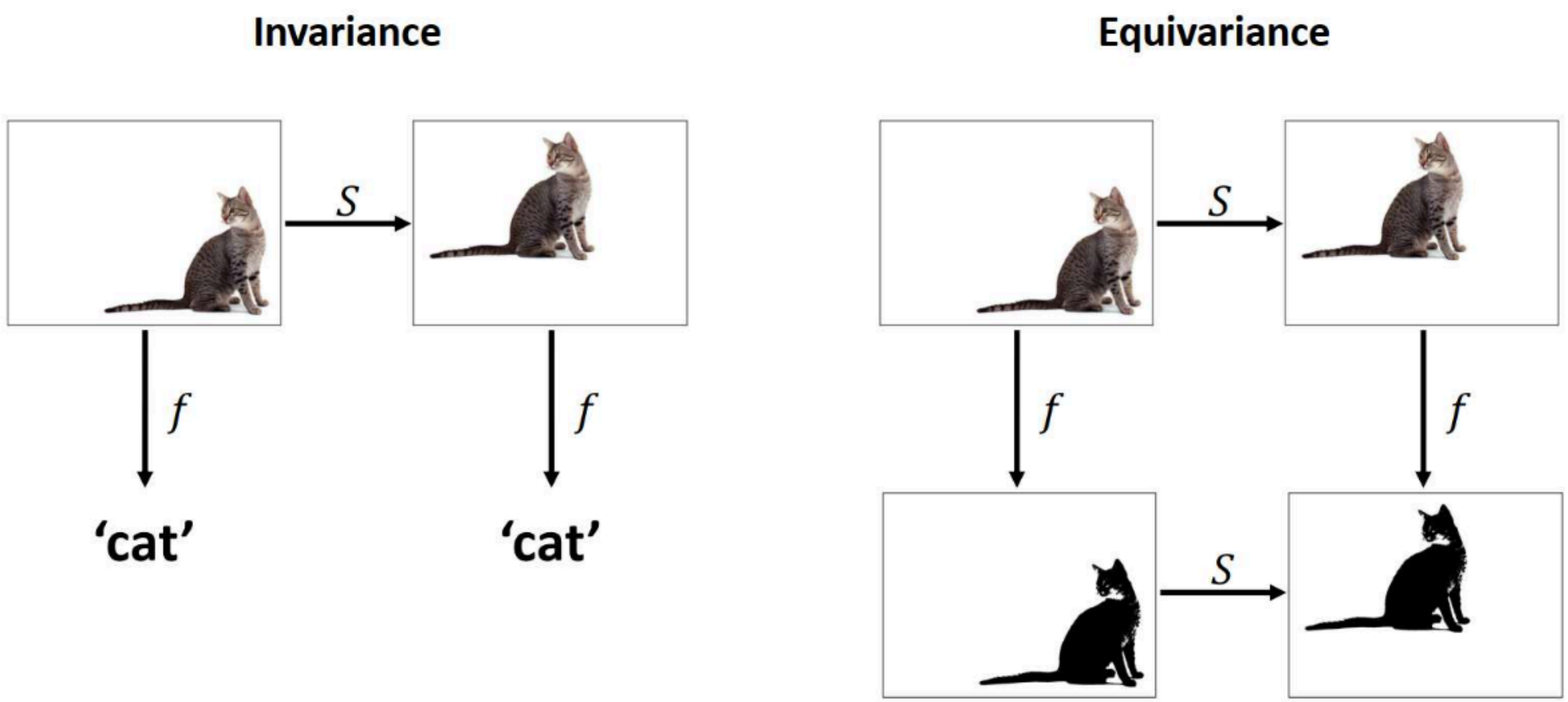
Output $f_i(\mathbf{x}) = \begin{cases} 1 & \text{pixel } i \in \text{cat} \\ 0 & \text{otherwise} \end{cases}$

Shifted input $S_v\mathbf{x}$

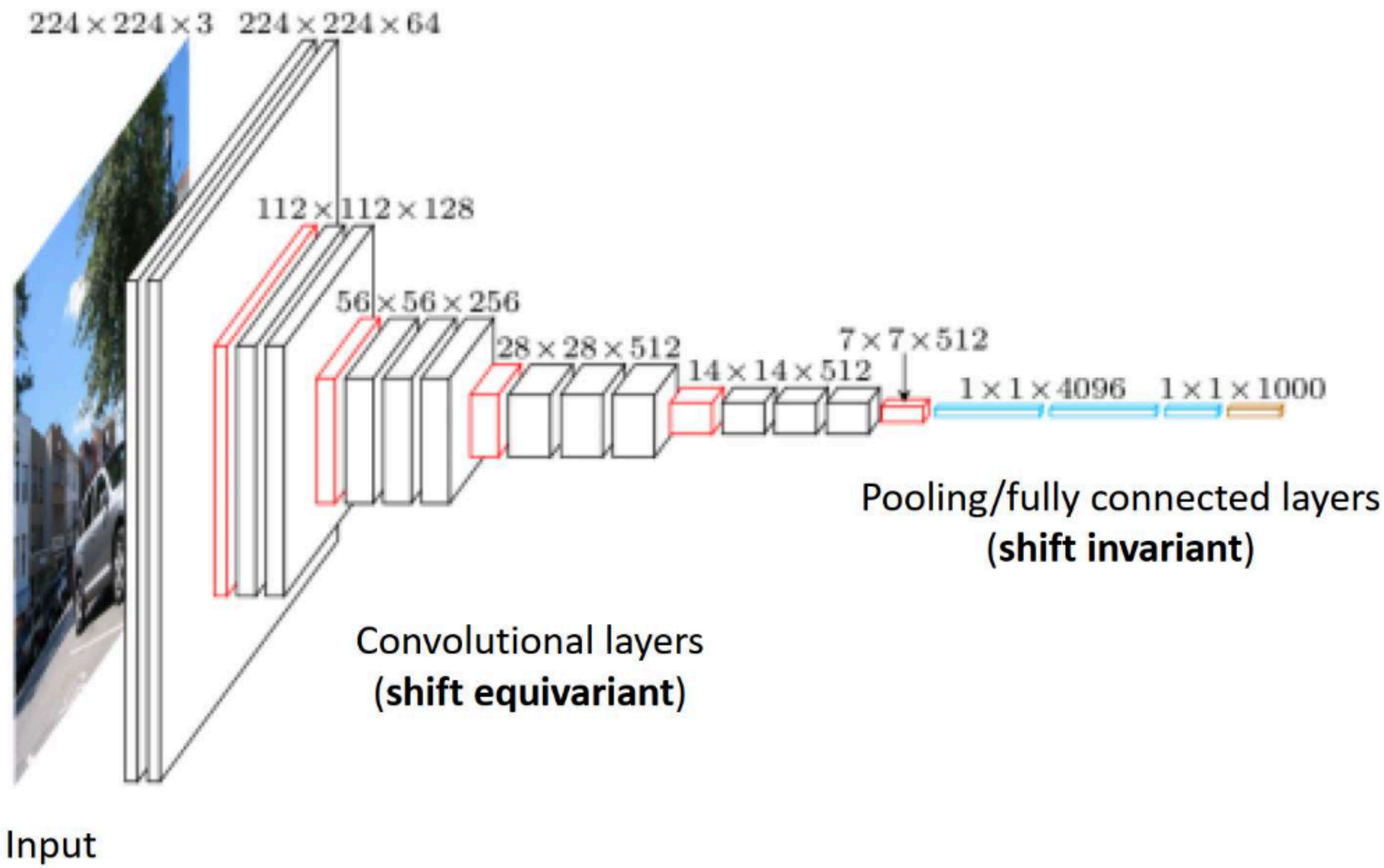


Output $f_i(S_v\mathbf{x}) = \begin{cases} 1 & \text{pixel } i \in \text{cat} \\ 0 & \text{otherwise} \end{cases}$

Invariance vs equivariance



How shift invariance is achieved in CNNs?



Dilated Convolutions

- Sometimes we want to have larger receptive fields in our networks
  - We can increase the kernel size to achieve this, but this introduces more weights
  - We can downsample/pool the input, but this decreases spatial resolution
  - Or we could ‘pad’ the kernel with zeros throughout to increase the effective size without increasing the number of parameters

Dilated Convolutions

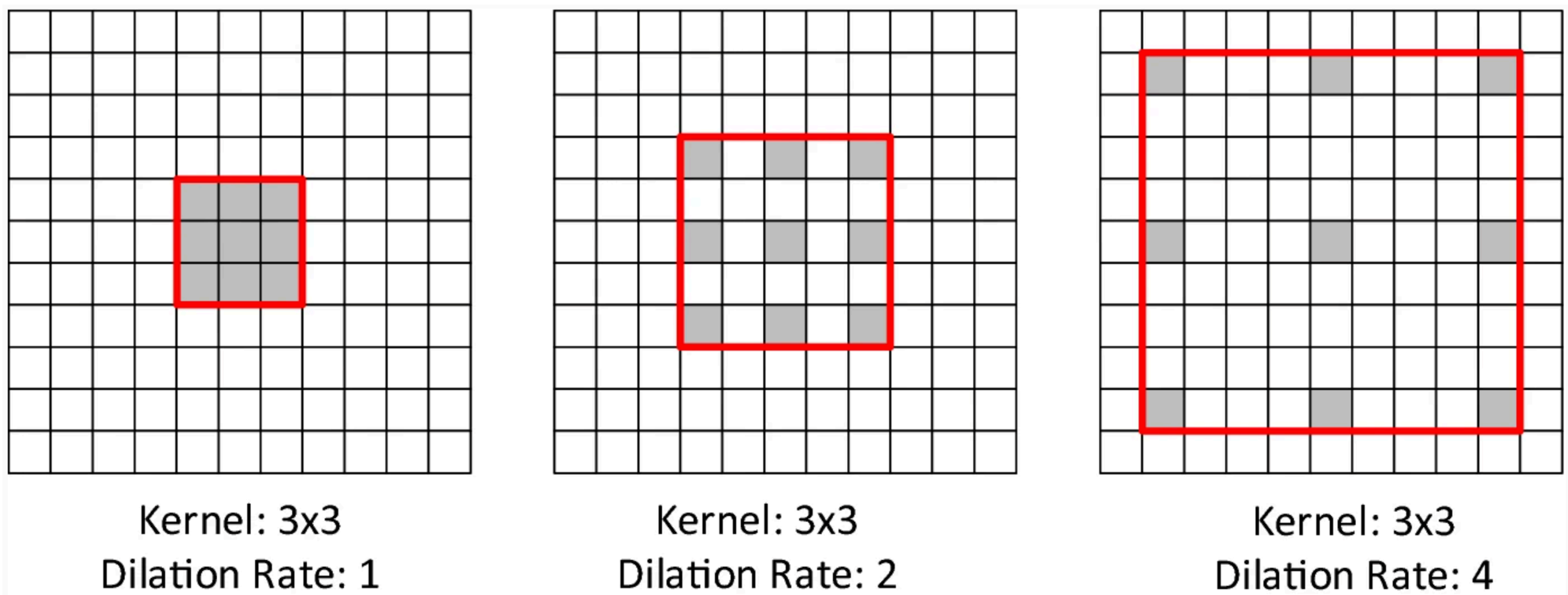


Image taken from https://link.springer.com/article/10.1007/s10618-021-00765-5/figures/1

Data Types

- Convolutions are applied to many dimensionalities and types of data - for example:

	Single Channel	Multichannel
1-D	Audio	Multiple sensor data over time
2-D	Audio data preprocessed into a spectrogram; greyscale images	Colour image data (e.g. RGB)
3-D	Volumetric data, e.g. CT scans	Colour video data